Stationary Configurations Imply Shift Symmetry: No Bondi Accretion for Quintessence / k-Essence.

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In this paper we show that, for general scalar fields, stationary configurations are possible for shift symmetric theories only. This symmetry with respect to constant translations in field space should either be manifest in the original field variables or reveal itself after an appropriate field redefinition. In particular this result implies that neither k-*Essence* nor *Quintessence* can have exact steady state / Bondi accretion onto Black Holes. We also discuss the role of field redefinitions in k-*Essence* theories. Here we study the transformation properties of observables and other variables in k-*Essence* and emphasize which of them are covariant under field redefinitions. Finally we find that stationary field configurations are necessarily linear in Killing time, provided that shift symmetry is realized in terms of these field variables.

I. INTRODUCTION

The discovery of the late time acceleration of the universe using Supernova Ia [1, 2] confirmed by other observations (see Ref. [3] and references therein), opened a window of opportunity for the existence of novel cosmological scalar fields not only during the early inflationary stage but also in the current universe. Indeed the scalar fields are the most natural candidates for realization of inflation and for the dynamical explanation of *Dark Energy* (DE) which is responsible for the late time acceleration. Arguably, the main difficulty in the modeling and understanding of the possible dynamics of *Dark Energy*, arises because of the fine tuning issues. In particular, there is the so-called coincidence problem [4, 5]: why the energy density in DE is only now comparable with the energy density in the dust-like *Dark Matter?* This coincidence would be especially remarkable, if one assumes that both these Dark constituents are independent of each other and evolve very differently in time. Partially because of the fine tuning problems it is not surprising that the candidates for DE often have not only rather exotic names: *Quintessence / Cosmon* [5, 6, 7, 8, 9, 10], k-*Essence* [11, 12, 13], *Phantom* [14], *Ghost Condensate* [15], *Quintom* [16] etc but also correspondingly very unusual properties. In particular, these scalar fields can possess: extremely small effective mass (*Quintessence*, *Quintom*), sound speed which can be much smaller and even larger than the speed of light (k-*Essence*, *Ghost Condensate*), negative kinetic energies (*Phantom*, *Quintom*), Lorentz symmetry breaking and gravity modifications even around the Minkowski space-time background (*Ghost Condensate*). The most successful paradigm to solve the coincidence problem is currently the k-*Essence*, where the highly nonlinear dynamics triggers the equation of state of DE from radiation-like to quasi de Sitter around the transition to the matter domination stage. In the late matter domination epoch the k-*Essence* has the speed of sound which is much smaller than one. However, it was showed [17, 18] that to explain the coincidence problem k-*Essence* models must necessarily have, at least, a short phase where the fluctuations in the k-*Essence* travel at superluminal speeds. For our paper it is important that the nonlinear dynamics responsible for the attractor behavior addressing the coincidence problem requires an explicit dependence of the Lagrangian on the field strength [18]. This field dependence cannot be eliminated by any field redefinitions. Thus, successful k-*Essence* models as well as *Quintessence / Cosmon* models cannot be shift symmetric.

On the other hand it is known that the current universe is highly inhomogeneous on small scales and in particular that there are plenty of Black Holes (BHs) of different mass and origin. Thus an interesting and natural question arises, how do Black Holes surrounded by cosmological scalar fields evolve? In addition, from the theoretical viewpoint it is interesting to consider BHs "dressed" with different field backgrounds. This could have a valuable impact on our understanding of the physics of horizons (see e.g. [19, 20, 21, 22]). Owing to the no-hair theorems [23, 24, 25, 26, 27] we know that BHs cannot support static configurations of scalar

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fields [1]. Therefore, any scalar hair will be continuously swallowed by the BH. In particular one could analyze the growth (and may be even formation) of Black Holes due to the accretion (collapse) of DE. Then one can try to use powerful and rather universal laws of Black Hole thermodynamics combined with astrophysical observations to restrict the allowed properties of DE candidates and rule out some of them as contradicting to either BH thermodynamics or astrophysical data. Recent studies along these lines were, for example, done e.g. in Ref. [19, 20, 21, 22, 29, 30, 31, 32, 33, 34, 35, 36].

Finally, for k-*Essence,* a typically very small sound speed during the late matter domination era allows for rather significant large-scale inhomogeneities around BHs and other massive objects. This long-range clumping would be one of the characteristic, potentially observable consequences of k-*Essence.* Moreover, due to this ability to realize small sound speeds along with the dust-like equation of state, the k-*Essence* fields can be used to model *Dark Matter* [37, 38, 39]. In this setup, the presence of supermassive BHs at the center of galaxies makes understanding the accretion process even more necessary.

On the other hand the presence of backgrounds with the superluminal sound speed mentioned above opens an exciting possibility to look beyond the BH horizon [20, 21][2]. Note that the current bounds [42, 43, 44, 45, 46] on DE sound speed are not restrictive at all.

The classical and most simple setup for accretion problems is a steady state or Bondi accretion [47]. Remarkably, a lot of astrophysical phenomena can be described by a steady state accretion. For a review see e.g. [48]. For scalar fields, the Bondi accretion was recently studied in [20, 21, 30, 31, 33, 49]. It is fair to say that almost all known analytical solutions [3] for accreting scalars either belong to the Bondi case or represent the dust-like free fall. The dust-like time dependent accretion of a massive canonical scalar field was considered in [30], while dust-like solutions for the *Ghost Condensate* scalars were found in [22, 35]. It seems that scalar fields with canonical kinetic terms would not leave any important impact on the astrophysical BHs in the current universe [30]. Nevertheless, accreting scalar fields could play an important role for the formation of primordial BHs (see e.g. [36]).

In this paper we investigate stationary configurations

for general k-*Essence* scalar field theories. We show that the necessary condition for the existence of exact stationary configurations is the symmetry of the theory with respect to constant shifts in the field space: $\phi \to \phi + c$. This symmetry has to be realized either in terms of the original field strength or after a field redefinition. On the way, we also analyze properties of general k-*Essence* scalar field theories covariant with respect to field redefinitions. The proof is valid for general theories with nonlinear kinetic terms in both the test-field approximation and the self-consistent case where the background metric is governed by the field ϕ itself. It is interesting to note that shift symmetric scalar field theories are exactly equivalent to perfect fluid hydrodynamics provided that only such field configurations which have timelike derivatives are considered. In particular this result implies that the most interesting scalar field models of *Dark Energy* cannot realize a steady state / Bondi accretion. Thus, in general, the solution to the problem of accretion of these fields onto Black Holes requires a knowledge of their initial configuration. In this paper we are discussing stationary configurations, which are exact. Of course for the real world the stationarity should be considered as an approximation. It may well happen that the solutions would only asymptotically approach the stationary regime. For some canonical scalar fields this behavior was demonstrated in [30].

II. DERIVATION OF THE STATIONARY CONFIGURATIONS

Let us consider a general scalar field theory with the action

$$ S = \int \mathrm{d}^4 x \sqrt{-g} P(\phi, X), \quad \text{where} \quad X = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi, $$

$g_{\mu\nu}$ is the gravitational metric and as usual $g \equiv \det g_{\mu\nu}$. Throughout the paper ∇_μ is the covariant derivative associated with the gravitational metric $g_{\mu\nu}$. We assume that the Lagrangian $P(\phi, X)$ is a general function satisfying the following conditions: $P_{,X} \geq 0$ (Null Energy Condition) and $2X P_{,XX}/P_{,X} > -1$ (Hyperbolicity condition) [4]. The first condition guaranties that the perturbations carry positive kinetic energy while the second one implies the stability with respect to high frequency perturbations and is necessary for the Cauchy problem to be well posed (see e.g. Refs. [37, 40, 50, 51, 52, 53]). These conditions restrict the variety of the allowed Lagrangians along with the corresponding solutions and are unavoidable for any physically meaningful model [5]. The

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[1] BHs can not support scalar hair at least for theories that respect some of the standard energy conditions. Having in mind the exotic properties of DE models mentioned above, it would be interesting to find examples of stable scalar hair in theories violating the usual energy conditions. For a model of hairy scalar BHs with ghost like quantum instabilities see Ref. [28].

[2] Despite of the presence of the superluminal propagation the accretion backgrounds constructed in these works are free of any causal pathologies [40]. However, it is interesting to study whether, similar to Ref. [41], two boosted BH could create causal paradoxes in this setup.

[3] See however Ref. [36]

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[4] In this paper we use the notation $(...)_{,X} \equiv \partial (...)/\partial X$ and the signature $(+ - - -)$.

[5] For a different opinion see [54].

energy-momentum tensor of the theory is

$$T_{\mu\nu} = P_{,X}\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}P. \qquad (1)$$

It is well known (see e.g. [55]), that for the timelike derivatives $X > 0$, the models under consideration can be described in a hydrodynamical language by introducing an effective four velocity [6]

$$u_\mu = \frac{\nabla_\mu\phi}{\sqrt{2X}}, \qquad (2)$$

along with the pressure

$$p = P(\phi, X), \qquad (3)$$

the energy density

$$\varepsilon(\phi, X) = 2XP_{,X} - P, \qquad (4)$$

and the sound speed [7]

$$c_s^2(\phi, X) = \left(1 + 2X\frac{P_{,XX}}{P_{,X}}\right)^{-1} = \left(\frac{\partial p}{\partial\varepsilon}\right)_\phi. \qquad (5)$$

In these variables the energy momentum tensor has the form corresponding to the one of a perfect fluid

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - pg_{\mu\nu}.$$

It is convenient to use the hydrodynamical notation for these functions of ϕ and X also for $X \leq 0$ when they do not have their usual physical meaning of velocity etc.

A. Field redefinitions and conditions for stationarity

If the field ϕ does not have any direct interactions except with gravity, then obviously a field redefinition $\phi = \phi(\tilde{\phi})$ cannot affect any observables besides the field itself. This is a particular case of a stronger statement (see e.g. Ref. [57]). Obviously the solutions $\phi(x)$ and $\tilde{\phi}(x)$ result through Einstein equations in the same gravitational metric $g_{\mu\nu}(x)$ and describe in that sense the same physical process. Thus it is interesting to investigate the properties of k-*Essence* under field redefinitions. Under field redefinitions $\phi = \phi(\tilde{\phi})$ we have

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[6] Note that even for $X > 0$ the effective four velocity introduced in (2) is not necessarily future directed. However, the analogy with the perfect fluid can be made exact by multiplying this expression (2) with ± 1 so that $u^0 > 0$. Furthermore, it is convenient to use the analytic definition of the square root so that every time when ϕ changes its sign the square root will change the sign as well preserving the future direction of u^μ.

[7] This formula for the sound speed was introduced for the cosmological perturbations in [56]. One can show [40] that the same expression is valid in the general case of backgrounds with timelike field derivatives: $X > 0$.

$\nabla_\mu\phi = \left(d\phi/d\tilde{\phi}\right)\nabla_\mu\tilde{\phi}$ whereas the expressions for the energy momentum tensor $T_{\mu\nu}$ and all hydrodynamical quantities ε, p, c_s and u^μ remain unchanged or covariant [8]. Here we should distinguish between covariance and invariance. Covariance means that the way how the quantities / equations are constructed from other objects remains unchanged whereas invariance implies exactly the same functional dependence on these objects. For example, the formula (4) defining the energy density ε through Lagrangian P, X and the derivative $P_{,X}$ looks the same after a field redefinition (covariant), however the dependence of the Lagrangian on the field does change (not invariant). It is obvious that e.g. the value of physical energy density at every point should not change under field redefinitions, but here these quantities reveal in addition such covariance with respect to the field redefinitions as it is the case for e.g. Euler-Lagrange equations. However, this covariance is not guarantied for all interesting objects. It is worthwhile mentioning that, e.g. the metric [40]

$$G_{\mu\nu}[\phi_0] = \left(\frac{P_{,X}}{c_s}\right)\left(g_{\mu\nu} - c_s^2\left(\frac{P_{,XX}}{P_{,X}}\right)\nabla_\mu\phi_0\nabla_\nu\phi_0\right),$$

describing the propagation of small perturbations π around a given background $\phi_0(x)$ transforms conformally under field redefinitions $\phi = \phi(\tilde{\phi})$:

$$G_{\mu\nu}[\phi_0] = \left(\frac{d\tilde{\phi}}{d\phi}\right)_0^2 G_{\mu\nu}[\tilde{\phi}_0].$$

Thus, as expected, the causal structure does not change under field redefinitions. The conformal factor $\left(d\tilde{\phi}/d\phi\right)_0^2$ compensates for the redefinition of perturbations $\pi = \left(d\phi/d\tilde{\phi}\right)_0\tilde{\pi}$.

Let us further consider a stationary space-time with metric $g_{\mu\nu}$ and a timelike Killing vector t^α. Thus $\pounds_t g_{\mu\nu} = 0$, where $\pounds_t$ is the Lie derivative. The configuration is stationary, if per definition

$$\pounds_t T_{\mu\nu} = 0.$$

Using Leibniz rule we have

$$\pounds_t T_{\mu\nu} = (\pounds_t P_{,X})\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\pounds_t P +$$
$$+P_{,X}\left[(\pounds_t\nabla_\mu\phi)\nabla_\nu\phi + (\pounds_t\nabla_\nu\phi)\nabla_\mu\phi\right] = 0 \qquad (6)$$

By multiplying this expression with $g^{\mu\nu}$ we obtain

$$0 = \pounds_t T_\mu^\mu = \pounds_t(2XP_{,X} - 4P) = \pounds_t(\varepsilon - 3p). \qquad (7)$$

Suppose the configuration $\phi(x^\mu)$ is such that $\nabla_\mu\phi$ is a null vector: $X = 0$. In that case we can multiply the right

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[8] Note that the four velocity (2) is invariant up to the sign only.

hand side of the Eq. (6) with $g^{\mu\nu}$ to obtain $\pounds_t P = 0$. Further we have $0 = \pounds_t P = P_{,\phi}\partial_t\phi$. As we are looking for stationary but not static solutions we have $P_{,\phi} = 0$. Thus the Lagrangian should be symmetric with respect to field shifts $\phi \to \phi + c$, where c is an arbitrary constant.

For $X \neq 0$ it is convenient to introduce the projector

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} - \frac{\nabla_\mu\phi\nabla_\nu\phi}{2X}, \tag{8}$$

with the properties

$$\mathcal{P}_{\mu\nu}\nabla^\nu\phi = 0, \ \ \mathcal{P}_{\mu\lambda}\mathcal{P}^{\lambda\nu} = \mathcal{P}_\mu^{\ \nu} \ \text{and} \ \ \mathcal{P}_\mu^\mu = 3. \tag{9}$$

Moreover, this projector is both invariant and covariant under field reparametrizations: $\mathcal{P}_{\mu\nu}[\phi] = \mathcal{P}_{\mu\nu}[\tilde{\phi}]$. By acting with the projector $\mathcal{P}^{\mu\nu}$ on the left hand side of Eq. (6) we have $0 = \mathcal{P}^{\mu\nu}\pounds_t T_{\mu\nu} = -3\pounds_t P$. Therefore, if the configuration is stationary then in particular

$$\pounds_t P = 0, \tag{10}$$

which for the hydrodynamical case reduces to the constancy of pressure p. Combining this with (7) we obtain the time independence of the energy density ε or

$$\pounds_t (XP_{,X}) = 0. \tag{11}$$

Further we can act on the left hand side of Eq. (6) with $\mathcal{P}^{\alpha\nu}$: so that $0 = \mathcal{P}^{\alpha\nu}\pounds_t T_{\mu\nu} = P_{,X}\mathcal{P}^{\alpha\nu}(\pounds_t\nabla_\nu\phi)\nabla_\mu\phi$. Thus, stationarity implies

$$\mathcal{P}^{\alpha\beta}(\pounds_t\nabla_\beta\phi) = 0.$$

Using the properties of the projector (9), Leibniz rule and that t^α is a Killing vector one obtains

$$0 = \mathcal{P}^{\alpha\beta}(\pounds_t\nabla_\beta\phi) = -\nabla_\beta\phi\pounds_t\mathcal{P}^{\alpha\beta} = -\pounds_t\nabla^\alpha\phi + \frac{\nabla^\alpha\phi}{2X}\pounds_t X.$$

The last expression in turn can be written in the following form

$$-\pounds_t\nabla^\alpha\phi + \frac{\nabla^\alpha\phi}{2X}\pounds_t X = \sqrt{2X}\pounds_t\left(\frac{\nabla^\alpha\phi}{\sqrt{2X}}\right).$$

Therefore, stationarity implies

$$\pounds_t\left(\frac{\nabla^\alpha\phi}{\sqrt{2X}}\right) = 0, \tag{12}$$

or in the hydrodynamical notation $\pounds_t u^\mu = 0$ [9]. Thus we have proved that for any stationary configuration the following conditions

$$\pounds_t u^\mu = 0, \ \ \pounds_t\varepsilon = 0 \ \text{and} \ \ \pounds_t p = 0, \tag{13}$$

should be satisfied. Note that these conditions are covariant under field redefinitions and, for the hydrodynamical case ($X > 0$), are intuitively clear requirements. Sometimes (see e.g. [31]) one claims that the stationarity implies a stronger requirement:

$$\pounds_t\nabla_\mu\phi = 0, \tag{14}$$

instead of the condition (12). However, the equation above is not covariant under the field redefinitions and does not follow from the stationarity of the energy momentum tensor.

Now let us find what type of theories $P(\phi, X)$ and field configurations $\phi(x^\mu)$ can, in principle, satisfy conditions (13). It is convenient to chose a coordinate system (t, x^i) such that the time coordinate corresponds to the integral curves of t^α. In that case the Lie derivative reduces to the partial derivative $\pounds_t = \partial_t$.

B. Which filed configurations can have constant effective four velocity u^μ ?

Now let us find the configurations $\phi(x^\mu)$ satisfying the condition on the effective four velocity (12). For the time component of the four velocity we have

$$\partial_t\left(\frac{\dot{\phi}}{\sqrt{2X}}\right) = \frac{\ddot{\phi}}{\sqrt{2X}} + \dot{\phi}\partial_t\left(\frac{1}{\sqrt{2X}}\right) = 0, \tag{15}$$

where $\dot{\phi} = \partial_t\phi$, while for the spatial components

$$\partial_t\left(\frac{\partial_i\phi}{\sqrt{2X}}\right) = \frac{\partial_i\dot{\phi}}{\sqrt{2X}} + \partial_i\phi\partial_t\left(\frac{1}{\sqrt{2X}}\right) = 0. \tag{16}$$

Obviously these equations have a trivial static solution $\phi = \phi(x^i)$. To find a nontrivial solution we combine these two equations to obtain following system of equations

$$\dot{\phi}\partial_i\dot{\phi} - \ddot{\phi}\partial_i\phi = 0,$$

which is equivalent to

$$\partial_t\left(\frac{\partial_i\phi}{\dot{\phi}}\right) = 0. \tag{17}$$

This is a system of partial differential equations of the second order. Integrating Eq. (17) we obtain the following linear homogeneous system

$$\partial_i\phi = V_i(x^j)\dot{\phi}, \tag{18}$$

where $V_i(x^j)$ are unknown time independent functions. This is the first order system of three partial differential equations for only one function ϕ. Let us find the consistency conditions under which the system can have solutions. Differentiating $i-$equation with respect to x^j

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[9] The vector u^μ is formally imaginary for $X < 0$. However, without any change of the results one could redefine u^μ in this case: $u^\mu = \nabla^\mu\phi/\sqrt{-2X}$.

and using the time differentiation of the $j-$equation we obtain

$$\partial_j \partial_i \phi = \partial_j V_i \dot\phi + V_i \partial_j \dot\phi = \partial_j V_i \dot\phi + V_i V_j \ddot\phi.$$

Now we can compare this result with the result of the same procedure performed for the $j-$equation. We obtain

$$\partial_i V_j - \partial_j V_i = 0.$$

For a simply connected manifold, the last equation implies the existence of a function (potential) $\Psi\left(x^i\right)$ such that $V_i = \partial_i \Psi$. Otherwise there are no solutions for (18).

For the $i-$equation we can assume that all x^k with $k \neq i$ are frozen parameters and for the characteristics (for the method of characteristics see e.g. excellent book [58]) we obtain

$$\frac{dt}{d\tau} = -\partial_i \Psi\left(x^j\right), \quad \frac{dx^i}{d\tau} = 1.$$

The first integral $\mathcal{I}$ of this system is given by the constant of integration for the equation

$$\frac{dt}{dx^i} = -\partial_i \Psi\left(x^j\right).$$

By integrating which we obtain

$$t = \mathcal{I} - \Psi\left(x^i\right),$$

therefore the general solution $\phi\left(t, x^i\right)$ is given as an arbitrary function of the first integral $\mathcal{I}$:

$$\phi\left(t, x^i\right) = \Phi\left(t + \Psi\left(x^i\right)\right). \tag{19}$$

Thus the general solution for equations (15) and (16) contains two arbitrary functions. Note that the system (17) does not have any other general solutions besides (19). It is easy to prove that this solution satisfies the equations (15) and (16). Indeed we have

$$\dot\phi = \frac{d\Phi}{d\mathcal{I}} \text{ and } \partial_i \phi = \frac{d\Phi}{d\mathcal{I}} \partial_i \Psi,$$

therefore

$$X = \frac{1}{2}\left(\frac{d\Phi}{d\mathcal{I}}\right)^2 \left(g^{00} + 2g^{0i}\partial_i \Psi + g^{ik}\partial_i \partial_k \Psi\right), \tag{20}$$

and the time component

$$\frac{\dot\phi}{\sqrt{2X}} = \frac{1}{\sqrt{g^{00} + 2g^{0i}\partial_i \Psi + g^{ik}\partial_i \partial_k \Psi}},$$

along with the spatial components

$$\frac{\partial_i \phi}{\sqrt{2X}} = \frac{\partial_i \Psi}{\sqrt{g^{00} + 2g^{0i}\partial_i \Psi + g^{ik}\partial_i \partial_k \Psi}},$$

are obviously time independent because the metric is stationary. It is worth mentioning that by using the condition (14) we would arrive at the general solution $\phi\left(t, x^i\right) = t + \Psi\left(x^i\right)$, missing the arbitrary functional dependence Φ. Note that arbitrary field redefinitions correspond to the freedom in choosing Φ.

C. Which Lagrangians do allow for the stationary configurations?

Now let us consider the restrictions on $P\left(\phi, X\right)$ arising from the requirement that the pressure and energy density should be time independent for the general solution (19). From Eq. (10) we have

$$\partial_t P = P_{,\phi}\dot\phi + P_{,X}\dot X = 0, \tag{21}$$

while from Eq. (11)

$$\partial_t \left(X P_{,X}\right) = \dot X P_{,X} + X P_{,X\phi}\dot\phi + X P_{,XX}\dot X = 0.$$

Eliminating $\dot X$ from these equations results in

$$X P_{,X\phi} - \left(X P_{,XX} + P_{,X}\right)\frac{P_{,\phi}}{P_{,X}} = 0. \tag{22}$$

This equation is a second order partial differential equation for $P\left(\phi, X\right)$. A trivial solution of this equation is a shift symmetric Lagrangian $P\left(X\right)$. It is well known that shift symmetric theories are exactly equivalent to hydrodynamics for $X > 0$. Obviously hydrodynamics usually allows for the steady flows. Let us find a general solution of the equation (22). This general solution should depend on two arbitrary functions. It is convenient to rewrite Eq. (22) in the following form

$$\frac{\partial \ln\left(P_{,\phi}/P_{,X}\right)}{\partial \ln X} = 1.$$

Integrating this equation we obtain

$$P_{,\phi} = \sigma\left(\phi\right) X P_{,X}, \tag{23}$$

where $\sigma\left(\phi\right)$ is an arbitrary function. The last equation (24) is a linear partial differential equation of the first order. Similarly to our previous calculations we use the method of characteristics to find the general solution. For the characteristics we have

$$\frac{d\phi}{d\tau} = 1 \text{ and } \frac{dX}{d\tau} = -\sigma\left(\phi\right) X, \tag{24}$$

thus the integral curves are given by the equation

$$\frac{dX}{d\phi} = -\sigma\left(\phi\right) X.$$

The general solution of the last equation is

$$X = \mathfrak{I}\exp\left(-\int \sigma\left(\phi\right)d\phi\right),$$

where $\mathfrak{I}$ is a constant of integration. Thus the general solution to the equations (23) and (22) is an arbitrary function of the first integral $\mathfrak{I}$ of the dynamical system (24):

$$P\left(\phi, X\right) = F\left(X e^{f(\phi)}\right), \tag{25}$$

where F and $f(\phi) = \int \sigma(\phi) d\phi$ are arbitrary functions. Note that all solutions of (22) are described by (25). It is obvious that the Lagrangian (25) has a hidden shift symmetry. Namely, we can always perform a field redefinition

$$\tilde{\phi}(\phi) = \int d\phi \, e^{f(\phi)/2}, \qquad (26)$$

so that the new Lagrangian is shift symmetric $P(\phi, X) = F(\tilde{X})$, where $\tilde{X} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \tilde{\phi} \nabla_\nu \tilde{\phi} = X e^{f(\phi)}$. Thus all scalar field theories which allow for stationary configurations are necessarily shift symmetric (explicitly or after field redefinition). Further, we will use the notation $\tilde{\phi}$ always for such field variables in which the system is invariant under shift transformations $\tilde{\phi} \to \tilde{\phi} + c$, where c is an arbitrary constant .

Finally we can specify the profiles Φ of stationary configurations. Equations (21) and (20) yield

$$P_{,\phi} + P_{,X} \left(\frac{d^2\Phi}{d\mathcal{I}^2} \right) \left(g^{00} + 2g^{0i} \partial_i \Psi + g^{ik} \partial_i \partial_k \Psi \right) = 0,$$

and using Eq. (23) and (20) we obtain

$$\frac{d^2\Phi}{d\mathcal{I}^2} + \frac{1}{2} \left(\frac{d\Phi}{d\mathcal{I}} \right)^2 \sigma(\Phi) = 0. \qquad (27)$$

We know that in terms of the new field $\tilde{\phi}$ the Lagrangian is shift symmetric. Thus for this parametrization $\sigma(\tilde{\phi}) = 0$. Therefore $\tilde{\Phi}(\mathcal{I}) = \alpha \mathcal{I} + \beta = t + \Psi(x^i)$ where we have absorbed the constants into Ψ and t. Thus in terms of the field variable $\tilde{\phi}$, in which the theory is shift symmetric, the possible stationary configurations are always given by

$$\tilde{\phi} = t + \Psi(x^i), \qquad (28)$$

and we are back to the usual ansatz (14). The stationary configurations in terms of the field variable ϕ can be obtained by solving equation (26) or (27) with respect to ϕ. This procedure determines the function Φ. While the function $\Psi(x^i)$ has to be fixed from the equations of motion and boundary/initial conditions.

It is worth noting that if the metric $g_{\mu\nu}$ possesses another Killing vector corresponding to e.g. axial symmetry $\mathcal{L}_\theta g_{\mu\nu} = 0$ then we can apply the result (28) to the angular variable θ. Thus the solution is

$$\tilde{\phi} = t + \Omega\theta + \Psi(x^i_\perp),$$

where Ω is a constant and $x^i_\perp$ denotes the rest of the coordinates.

III. CONCLUSIONS AND DISCUSSION

In this paper we have proved that the existence of stationary configurations requires shift symmetry. Namely

(may be after a field redefinition) the system has to be invariant with respect to the transformation $\tilde{\phi} \to \tilde{\phi} + c$, for all constants c. The result is valid in the self-consistent case where the geometry is produced by the scalar field as well as in the test field approximation where the stationary field configuration appears on the gravitational background governed by other sources. The shift symmetry implies the conservation of the Noether current

$$J_\mu = P_{,\tilde{X}} \nabla_\mu \tilde{\phi}.$$

Interestingly, the equation of motion implies $\nabla_\mu J^\mu = 0$, which is a statement of the conservation of the current J^μ. In the case when $\nabla_\mu \tilde{\phi}$ is timelike the current J^μ can be written in the form of an effective particle density current $J^\mu = \tilde{n} u^\mu$, where the particle density [10] is

$$\tilde{n} = \sqrt{2\tilde{X}} P_{,\tilde{X}}.$$

Note that this current is not covariant under field redefinitions. The conservation $\nabla_\mu J^\mu = 0$ of the particle density current usually holds in the standard hydrodynamics. However, the most interesting models of cosmological scalar fields do not posses this additional conservation law associated with the shift symmetry. Thus the result obtained in this paper implies that there is no exact Bondi (steady flow) accretion for popular classes of models for dynamical *Dark Energy* like *Quintessence* and k-*Essence*. This result may not have a very strong qualitative impact on the growth of Black Holes or on the evolution of the cosmological fields around them. Indeed, one should expect that the accretion rate should be in any case rather small (for the case of canonical scalars see Ref. [30]) . Especially in the late / current universe, one can almost always neglect the growth of the Black Hole along with the corresponding backreaction. Nevertheless, this result changes the setup for the investigation of the problem. Now in order to study how these fields could accrete onto Black Holes one is forced to solve the Cauchy problem for nonlinear partial differential equations, instead of solving the boundary problem for nonlinear ordinary differential equations. In particular to approach this problem one has to choose some initial configuration for the field and it's time derivative. At this stage, it is not clear what are reasonable, physically motivated initial conditions and at what time they should be posed. This is very different from the case of Bondi accretion where the boundary conditions are fixed by cosmological evolution and the membrane property of the BH horizon. However, it may happen that there are some special attractor or self-similar regimes to which the solutions would approach in the late time asymptotic. Nevertheless, one cannot guarantee either the existence of these attractors nor their uniqueness for

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[10] Note that this number density is none other than the canonical momenta for the field $\tilde{\phi}$ in the co-moving reference frame.

a general model. Moreover, even if a unique attractor exists, then it is not a priory known how wide the base of attraction is in the phase space consisting of initial configurations of the field and its time derivative. Thus, the procedure for finding these attractor solutions is not only a predominantly numerical exercise, but also generically not very promising and predictive. Nevertheless, it is very interesting to find examples of scalar field systems possessing solutions of this type. In [30] it was demonstrated that for canonical scalars and many potentials the solutions indeed approach the steady flow.

In addition one has to mention that having a shift symmetric theory is a necessary, but not sufficient for the existence of a stationary configuration. For example, in hydrodynamics there can be either exceptional theories or even exceptional boundary conditions for which there are no stationary configurations. In particular the simple accretion of dust onto a Black Hole occurs along geodesics and therefore is not steady. A similar situation happens in the case of the *Ghost Condensate* for which the accretion rate blows up when the field configuration at spatial infinity approaches the condensation point (compare Refs. [31] and [35]). Moreover, in the DBI model considered in [20, 21] it was found that a physically meaningful steady state accretion is not possible when the sound speed at spatial infinity is $c_s^2 > 4/3$.

In this paper we have considered only a single self interacting scalar field. It would be interesting to study other types of fields, in particular one could think of scalars with internal degrees of freedom e.g charged scalars accreting onto a charged Black Hole. We expect that the appearance of new external forces and internal degrees of freedom can change the picture. Another interesting problem is to find possible attractor or self-similar asymptotic solutions and develop a perturbation theory around them. As we have shown stationary configurations are possible only for theories which are equivalent to perfect fluids. This result reveals once again that the relation between hydrodynamics and field theory is rather deep. Therefore we think this connection deserves a further study. We found that investigation of possible dynamical backgrounds around Black Holes is interesting not only from the point of view of mathematical physics but may be relevant for a better understanding of both Black Holes physics and may be even the nature of *Dark Energy*.

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